The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

August 2025

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let V be a finite-dimensional real inner product space. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V that is invariant under T.

   (a) Show that $U^{\perp}$ is invariant under T^*.

   (b) Construct an example of a $T \in \mathcal{L}(V)$ with a subspace U for which U is invariant under T but $U^{\perp}$ is not invariant under T. In your answer, give V, T and U, then show that U is invariant under T and show that $U^{\perp}$ is not invariant under T.

2. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$.

   (a) (2.5 pts) Prove that T is invertible.

   (b) (7.5 pts) Find the minimal polynomial of T^{-1}.

3. (a) For each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^3$, assign a scalar $(\mathbf{x}, \mathbf{y})$ as follows:

$$(\mathbf{x}, \mathbf{y}) = \mathbf{y}^* \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \mathbf{x}.$$

   where $\mathbf{y}^*$ is the conjugate transpose of $\mathbf{y}$. Is $(\cdot, \cdot)$ an inner product on $\mathbb{C}^3$?

   (b) Let V be an inner product space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove or disprove

   (a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u} + \mathbf{w}\| + \|\mathbf{w} + \mathbf{v}\|$;

   (b) $| < \mathbf{u}, \mathbf{v} > | \leq | < \mathbf{u}, \mathbf{w} > | + | < \mathbf{w}, \mathbf{v} > |$.

4. Let T be a linear operator from $\mathbb{R}^5$ to $\mathbb{R}^5$ defined by

$$T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e).$$

   (a) Find the characteristic and minimal polynomial of T.

   (b) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.

   (c) Find the Jordan form of T with respect to your basis.

5. Let $A \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$ be a unit vector in the 2-norm, $\tau \in \mathbb{R}$ and $\mathbf{r} = A\mathbf{x} - \tau\mathbf{x}$.

(a) Show that τ is an eigenvalue of a matrix $A + E$, where $\|E\|_2 \leq \|\mathbf{r}\|_2$.

(b) Assuming in addition that A is symmetric, show that there exists an eigenvalue λ of A such that $|\lambda - \tau| \leq \|\mathbf{r}\|_2$.

6. (a) Let $x, y \in \mathbb{R}^n$ such that $x \neq y$ but $\|x\|_2 = \|y\|_2$, show that there exists a reflector Q of the form $Q = I - 2uu^T$, where I is the $n \times n$ identity matrix, $u \in \mathbb{R}^n$, and $\|u\|_2 = 1$ such that $Qx = y$.

   (b) Let $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$, compute a reduced QR decomposition of A using Householder reflections and then solve the least square problem $\min_x \|b - Ax\|_2$ and calculate error $\|b - Ax\|_2$.

7. Let $A = QTQ^*$ be a Schur decomposition of the matrix

$$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}.$$

   Find such a matrix T.

8. (a) Suppose $p, q \in \mathbb{R}$ with p and q positive and $1/p + 1/q = 1$. Show that for any matrix $A \in \mathbb{C}^{n\times n}$, we have $\|A\|_p = \|A^*\|_q$, where A^* is the conjugate transpose of A. Here $\|A\|_p$ denotes the matrix $p-$norm induced by the vector p-norm defined by $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$.

   (b) Prove that

$$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$

   for any $A \in \mathbb{C}^{n\times n}$ and any positive p and $q \in \mathbb{R}$ with $1/p + 1/q = 1$.

   (c) Prove that for any $p \geq 1$ and any diagonal matrix $D \in \mathbb{C}^{n\times n}$, we have

$$\|D\|_p = \max\{|d_{ii}| : 1 \leq i \leq n\}\,.$$

   (d) Show that $\|A\|_2$ is the largest singular value of A.