Joint Program Exam in Mathematical Analysis
May 12, 2025

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.

1. (i) Let $\sigma > 1$. Show that if a sequence $\{a_n\}$ satisfies

$$|a_n - a_{n-1}| \leq \frac{C}{n^\sigma},$$

then $\{a_n\}$ is convergent.

(ii) Let $0 < p < 1$. Show that the sequence

$$a_n = 1 + \frac{1}{2^p} + \ldots + \frac{1}{n^p} - \frac{n^{1-p}}{1-p}$$

is convergent.

2. Let $f, g : [0,1] \to \mathbb{R}$ be two continuous functions, so that $\sup_{x\in[0,1]} f(x) = \sup_{x\in[0,1]} g(x)$. Prove that there exists $x_0 \in [0,1]$, so that $f(x_0) = g(x_0)$.

3. Let $g : [a,b] \to R$ be continuously differentiable. Show that

$$\max_{x\in[a,b]} |g(x)| \leq \frac{1}{b-a}\int_a^b |g(x)|\,dx + \int_a^b |g'(x)|\,dx.$$

4. Let $f : [0,1] \to \mathbf{R}$ be a twice differentiable function, with $M = \sup_{0<x<1} |f''(x)|$. Prove that for every positive integer N,

$$\left| \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{k=0}^{N-1} f\left(\frac{k+1/2}{N}\right) \right| \leq \frac{M}{24N^2}.$$

5. Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{x^2+n}}$$

converges uniformly on $[-M, M]$ for all $M > 0$.

6. Assume that for $n \in \mathbb{N}$, g_n is Riemann integrable on $[0,1]$, $g_n \to g$ pointwise on $[0,1]$ and uniformly on any subinterval $[0,b]$ for $0 < b < 1$, and for some constant $M > 0$, $|g_n(x)| \leq M$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Show that g is Riemann integrable on $[0,1]$, and

$$\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = \int_0^1 g(x)\,dx.$$

7. Let $K : [0,1]^2 \to \mathbb{R}$ be a continuous function such that $|K(x,y)| < 1$ for all $(x,y) \in [0,1]^2$. Let $g : [0,1] \to \mathbb{R}$ be a continuous function.

(i) Let $u : [0,1] \to \mathbb{R}$ be a continuous function. Show that the function

$$\psi(x) := \int_0^1 K(x,y)u(y)\,dy + g(x), \qquad x \in [0,1],$$

is well defined and continuous on $[0,1]$.

(ii) Use the contraction mapping theorem to show that the integral equation

$$u(x) = \int_0^1 K(x,y)u(y)\,dy + g(x), \qquad x \in [0,1],$$

has an unique solution that is continuous on $[0,1]$.