NAME:________________________________

GRADE:________________________________

SCHOOL NAME:________________________

P	1	2	3	4	5	6	7	Total
S								

2024-2025 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with correct justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (10 pts) A group of 2024 people are lifting weights. The first one lifts 100 pounds, the second one lifts 80 pounds, the third one lifts 120 pounds, and from this moment on every next person lifts the average of the weights that all the people before this person lifted. What weight was lifted by the 2024-th person?

YOUR ANSWER:

PROBLEM 2 (20 pts) In a math club run by cats and dogs, cats form more than 40% but less than 50% of the participants. What is the minimal overall number of animals in such a club?

YOUR ANSWER:

PROBLEM 3 (30 pts) Student A and student B calculated the product of two integers times their difference. A claims that the result is 75,139 while B says it is 75,144. It is known that only one of them is correct. Who?

YOUR ANSWER:

over, please

PROBLEM 4 (40 pts) Compute out the number

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})\cdots(1-\frac{1}{21^2}).$$

YOUR ANSWER:

PROBLEM 5 (80 pts) There are several containers each of which weighs at most 1 ton. Their weights add up to 10 tons. A company needs to bring them from Birmingham to Atlanta and has trucks each of which can carry 3 tons. What is the minimal number of trucks that the company needs **to guarantee** that the job will be done regardless of how the weights are distributed among the containers? Each truck is supposed to make one trip from Birmingham to Atlanta.

YOUR ANSWER:

PROBLEM 6 (110 pts) A quadrilateral ABCD is inscribed in a circle. Its diagonals AC and BD are perpendicular. The straight line that is perpendicular to AD and passes through B intersects AC at a point X. The straight line that is perpendicular to AD and passes through C intersects BD at a point Y. It is given that $|BC| = 12$. Find $|BX| + |XY| + |YC|$.

YOUR ANSWER:

PROBLEM 7 (160 pts) A seller on a market sells 20 watermelons that weigh 1, 2, 3, ..., 20 lbs. He is using k pairs of equal weights (which weigh $x_1, x_1; x_2, x_2; \ldots; x_k, x_k$ pounds, with all x_i's being distinct) and manages to weigh each watermelon with these weights precisely (as he weighs watermelons, he puts at most two weights on one side of the balance and a watermelon on the other side, thus making them of equal weight). What is the minimal k for which it is possible?

YOUR ANSWER:

2024-2025 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) A group of 2024 people are lifting weights. The first one lifts 100 pounds, the second one lifts 80 pounds, the third one lifts 120 pounds, and from this moment on every next person lifts the average of the weights that all the people before this person lifted. What weight was lifted by the 2024-th person?

YOUR ANSWER: **100 pounds**.

Solution: If k people lifted weights so that the average weight they lifted is x pounds, and then the $k+1$-st person lifts x pounds, then the average weight for the first $k+1$ people remains x pounds. Since the first people lifted 300 pounds, the average from the 4-th person on remains 100 pounds.

The answer is **100 pounds**. □

PROBLEM 2 (20 pts) In a math club run by cats and dogs, cats form more than 40% but less than 50% of the participants. What is the minimal overall number of animals in such a club?

YOUR ANSWER: **7.**

Solution: If c is the number of cats, d is the number of dogs, and $N = c + d$ is the overall number of animals in the club, then the following inequalities hold: $2/5 < c/N < 1/2$. Since all numbers involved are positive integers, it is easy to see that the smallest N for which it is possible is 7; then we can set $c = 3$, and have $2/5 < 3/7 < 1/2$ as desired.

The answer is **7**. □

PROBLEM 3 (30 pts) Student A and student B calculated the product of two integers times their difference. A claims that the result is 75,139 while B says it is 75,144. It is known that only one of them is correct. Who gave the correct answer, and what it was?

YOUR ANSWER:

Solution: If the two integers in question are n and m, then the students are computing out the number $mn(m - n) = m^2n - mn^2$. Evidently, if both m or n are odd then both m^2n and mn^2 are odd; otherwise both m^2n and mn^2 are even. Therefore $mn(m-n)$ is always even. Hence the correct answer is 75,144, it is B who was correct. In fact, $75,144 = 101 \cdot 93 \cdot (101 - 93)$.

The answer is **B; 75,144**. □

PROBLEM 4 (40 pts) Compute out the number

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})\cdots(1-\frac{1}{21^2}).$$

YOUR ANSWER: $\mathbf{\frac{11}{21}}$.

Solution: Let us simplify the product in question as follows:

$$(1-\frac{1}{2^2})\cdot(1-\frac{1}{3^2})\cdot(1-\frac{1}{4^2})\dots(1-\frac{1}{21^2}) =$$
$$= (1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{3})\cdots(1-\frac{1}{21})(1+\frac{1}{21}) =$$
$$= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdots\frac{20}{21}\cdot\frac{22}{21}.$$

In other words, on an intermediate step the following cancellation takes place:

$$(1-\frac{1}{n^2})(1-\frac{1}{(n+1)^2}) = (1-\frac{1}{n})(1+\frac{1}{n})(1-\frac{1}{n+1})(1+\frac{1}{n+1}) =$$
$$= \frac{n-1}{n}\cdot\frac{n+1}{n}\cdot\frac{n}{n+1}\cdot\frac{n+2}{n+1}$$

which implies that all fractions in the middle of the product will get cancelled. The remaining two fractions that will not be cancelled are, evidently, $\frac{1}{2}$ and $\frac{22}{21}$, so the answer is their product.

The answer is $\mathbf{\frac{11}{21}}$. □

PROBLEM 5 (80 pts) There are several containers each of which weighs at most 1 ton. Their weights add up to 10 tons. A company needs to bring them from Birmingham to Atlanta and has trucks each of which can carry 3 tons. What is the minimal number of trucks that the company needs **to guarantee** that the job will be done regardless of how the weights are distributed among the containers? Each truck is supposed to make one trip from Birmingham to Atlanta.

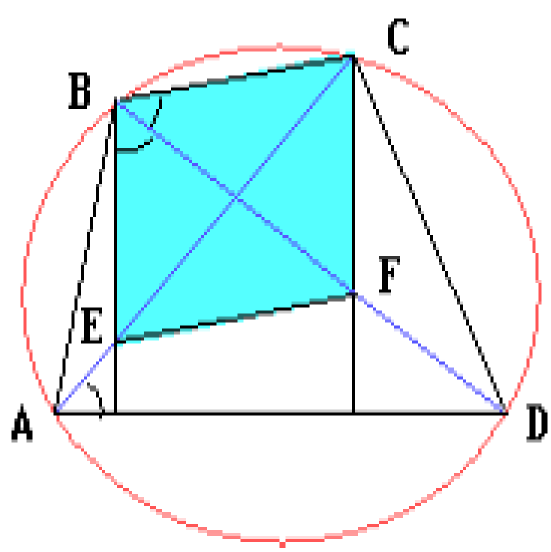
YOUR ANSWER: **5 trucks.**

Solution: Let us prove that 5 trucks is enough. Indeed, we can load boxes in truck in any order stopping only if we see that no matter which next box we try to load the truck will be overloaded. Since each box is at most 1 ton, each truck will be loaded with at least 2 tons of cargo until all remaining boxes weigh less than 3 tons when we will load the last truck with the remaining boxes. This implies that the first 4 truck will be loaded with at least 8 tons overall after which the 5-truck will take care of the remaining boxes. Thus, 5 trucks can carry the entire cargo of 10 tons from Birmingham to Atlanta.

But maybe 4 trucks will work? No, they will not!

Indeed, let us show that 4 trucks are not enough. Suppose that there are 13 boxes, 10/13 tons each. Then a truck can carry at most 3 boxes because 4 boxes weigh $40/13 > 3$ tons. Hence 4 trucks will only be able to carry 12 boxes.

The answer is **5 trucks.** □



PROBLEM 6 (110 pts) A quadrilateral ABCD is inscribed in a circle. Its diagonals AC and BD are perpendicular. The straight line that is perpendicular to AD and passes through B intersects AC at a point E. The straight line that is perpendicular to AD and passes through C intersects BD at a point F. It is given that $|BC| = 12$. Find $|BE| + |EF| + |FC|$.

YOUR ANSWER: **36.**

Solution: Set $\angle DBC = \alpha, \angle ACB = \beta$. Clearly, $\alpha + \beta = \pi/2$. Then properties of angles imply that $\angle CAD = \alpha$, hence $\angle BEC = \beta$ and $|EQ| = |QC|$ where Q is the point of intersection of EC and BF. Similarly, $\angle BDA = \beta$, hence $\angle BFC = \alpha$, hence $|BQ| = |QF|$. We conclude that $EBCF$ is a rhombus and $|BE| + |EF| + |FC| = 36$.

The answer is **36**. □

PROBLEM 7 (160 pts) A seller on a market sells 20 watermelons that weigh 1, 2, 3, ..., 20 lbs. He is using k pairs of equal weights (which weigh $x_1, x_1; x_2, x_2; \ldots; x_k, x_k$ pounds, with all x_i's being distinct) and manages to weigh each watermelon with these weights precisely (as he weighs watermelons, he puts at most two weights on one side of the balance and a watermelon on the other side, thus making them of equal weight). What is the minimal k for which it is possible?

YOUR ANSWER: **6**.

Solution: 6 weights of 1, 3, 5, 7, 9 and 10 are sufficient: 1=1, 2 = 1 + 1, 4 = 3 + 1, 6 = 5 + 1, 8 = 7 + 1, 11 = 10 + 1, 12 = 9 + 3, 13 = 10 + 3, 14 = 9 + 5, 15 = 10 + 5, 16 = 9 + 7, 17 = 10 + 7, 18 = 9 + 9, 19 = 10 + 9, 20 = 10 + 10.

But maybe 5 weights is enough? No, it is not!

Indeed, let us show that 5 weights are not enough. Suppose that we can do the job with 5 weights. Then *each* combination of one or two weights gives us a distinct number between 1 and 20 (there are 5 one-weight combinations, 5 "two equal weights" combinations, and 10 "two distinct weights" combinations). In particular, this implies that each weight is an integer. Now, assume that there are k odd weights and $5 - k$ even weights for some k. Then the number of combinations of weight that produce an odd sum is 10 and, on the other hand, can we expressed as $k + k(5 - k)$. Thus, $k + k(5 - k) = 10$ and we get an equation $k^2 - 6k + 10 = 0$ which has no real solutions.

The answer is **6**. $\square$